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Contact mechanics

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Abstract

Contact problems are central to Solid Mechanics, because contact is the principal method of applying loads to a deformable body and the resulting stress concentration is often the most critical point in the body. Contact is characterized by unilateral inequalities, describing the physical impossibility of tensile contact tractions (except under special circumstances) and of material interpenetration. Additional inequalities and/or non-linearities are introduced when friction laws are taken into account. These complex boundary conditions can lead to problems with existence and uniqueness of quasi-static solution and to lack of convergence of numerical algorithms. In frictional problems, there can also be lack of stability, leading to stick–slip motion and frictional vibrations.

If the material is non-linear, the solution of contact problems is greatly complicated, but recent work has shown that indentation of a power-law material by a power law punch is self-similar, even in the presence of friction, so that the complete history of loading in such cases can be described by the (usually numerical) solution of a single problem.

Real contacting surfaces are rough, leading to the concentration of contact in a cluster of microscopic actual contact areas. This affects the conduction of heat and electricity across the interface as well as the mechanical contact process. Adequate description of such systems requires a random process or statistical treatment and recent results suggest that surfaces possess fractal properties that can be used to obtain a more efficient mathematical characterization.

Contact problems are very sensitive to minor profile changes in the contacting bodies and hence are also affected by thermoelastic distortion. Important applications include cases where non-uniform temperatures are associated with frictional heating or the conduction of heat across a non-uniform interface. The resulting coupled thermomechanical problem can be unstable, leading to a rich range of physical phenomena.

Other recent developments are also briefly surveyed, including examples of anisotropic materials, elastodynamic problems and fretting fatigue.

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A casual survey of the kinds of engineering applications to which the techniques of Solid Mechanics are applied will show that the vast majority of solid bodies are loaded by being pressed against another body. The only alternatives comprise loading of the boundary by fluid pressure or various kinds of body force such as gravitational or magnetic forces, but even in such cases, the reaction force required to maintain equilibrium will almost invariably be provided at a contact interface. When we also recall that contacts between bodies generally constitute stress concentrations and are therefore likely sites for material failure, it is not surprising that Contact Mechanics has occupied a central place in the development of Solid Mechanics over the years and continues to do so today.

Additional interest in the subject is generated by the fact that the inevitable roughness of contacting surfaces generates a very complex local structure at which extreme conditions are likely to occur, particularly if sliding takes place, leading to frictional heating and very high local temperatures.

Historically, the development of the subject stems from the famous paper of Heinrich Hertz (1882) giving the solution for the frictionless contact of two elastic bodies of ellipsoidal profile. Hertz' analysis still forms the basis of the design procedures used in many industrial situations involving elastic contact. Since 1882, the subject of contact mechanics has seen considerable development. Two major threads can be distinguished — from a mathematical standpoint, emphasis has been placed on the extension of Hertz' analysis to other geometries and constitutive laws and on the proof of theorems of existence and uniqueness of solution, whereas engineers have focussed on the solution of particular problems in an attempt to understand and influence phenomena that occur in practical contacting systems, both on the macro and the micro scale.

Gladwell (1980) provides a compendious treatment of the various contact geometries that had been treated up to that time, including an invaluable survey of the rich Russian literature on the subject. Johnson (1985) gives an excellent overview of the range of contact problems that have come under consideration and achieves a nice balance between mathematical rigour and engineering practicality. In an attempt to obtain a similar balance, we first revisit the defining characteristics of contact mechanics in the mathematical framework of problems involving unilateral inequalities. Particular attention is given to the additional features associated with the presence of friction at the interface, where non-existence, non-uniqueness and instability of the quasi-static solution can be obtained with sufficiently high friction coefficients. We then introduce the concept of self-similarity, which provides a powerful method for indentation problems even for non-linear materials and other aspects of contact beyond the elastic limit are also discussed. The specialized areas of anisotropic and elastodynamic contact are briefly summarized and the paper concludes with a discussion of recent developments in the characterization and contact of rough surfaces and in thermoelastic contact.

1. Unilateral inequalities

The essence of a contact problem lies in the fact that any point on the boundary of each body must either be in contact or not in contact. If it is not in contact, the gap g between it and the other body must be positive ($g > 0$), whereas if it is in contact, $g = 0$, by definition. A dual relation involves the contact pressure p between the bodies which must be positive ($p > 0$) where there is contact and zero where there is no contact. The inequalities serve to determine which points will be in contact and which not. If the contact area is prescribed, it can be shown from classical existence and uniqueness proofs that the equations alone are sufficient to define the stresses and displacements throughout the bodies, but there is then of course no guarantee that the solution will satisfy the inequalities. Fichera (1964, 1972) proved that the complete problem, including the inequalities, has a unique solution when the material is linear elastic and many related proofs have since been advanced for other classes of contact problem (Duvaut and Lions, 1976). In problems with linear kinematics and smoothly turning boundaries, the inequalities imply that the contact tractions will tend to zero at the edge of the contact region. This can be demonstrated by examining the asymptotic fields at the transition between a region of contact and separation (Johnson, 1985: §5.1). Many authors use this condition of continuity of contact tractions in place of the inequalities, but it should be emphasised that the inequality formulation is the correct physical statement of the problem and indeed it can be shown that continuity of tractions is a necessary but not sufficient condition in certain classes of problem, particularly those leading to multiple contact areas.

Analytical solutions can be obtained only to a very limited class of contact problems and hence there has been considerable development of numerical methods. Algorithms to handle the contact inequalities are now routinely included in most commercial finite element packages, usually based on an appropriate smoothing of the discontinuities involved or the Lagrange multiplier technique. Apart from the unilateral boundary conditions, contact problems present difficulties because of the often very localized nature of the contact and the fact that the extent and location of the contact area changes during loading, which makes it difficult to choose an appropriate mesh. Discussion of the extensive literature on numerical algorithms for contact problems is beyond the scope of this review, but interested readers are referred to Kikuchi and Oden (1988), Klarbring (1993) and Zhong (1993). Algorithms for rolling contact problems, in particular for wheel/rail contact, are discussed by Kalker (1990).

1.1. Frictional problems

When there is friction at the contact interface, additional conditions are introduced, which considerably complicate the proofs of existence and uniqueness. The simplest friction condition is that of 'Coulomb' friction, according to which, any point in the contact area must be in one of two states — 'stick', during which there is no relative motion and the resultant tangential traction is less than fp, where f is the coefficient of friction, and 'slip', during which there is relative motion and the tangential traction is of magnitude fp and opposes the instantaneous direction of slip. Notice that frictional slip is essentially an incremental process and hence the solution depends on the history of loading. If the loading rate is sufficiently slow, it is possible to define a *quasi-static* solution in which inertia effects are negligible and the system passes through a sequence of equilibrium states. This in turn can often be reduced to an equivalent *static* problem when the loading is monotonic and proportional. Comninou and Dundurs (1982) discuss some simple examples which illustrate the complexity of history dependence which can arise from the apparent simplicity of the Coulomb friction conditions.

A classical example of the way in which Coulomb frictional tractions depend upon the history of loading was provided by Cattaneo (1938) who solved the problem of a normal Hertzian contact between similar materials followed by a monotonically increasing tangential force at constant normal force. Cattaneo showed that slip would occur in an elliptical annulus homothetic to the contact area and that the resulting frictional traction distribution would be the difference between the traction distribution at limiting friction and an opposing self-similar distribution in the central ellipse. Cattaneo's results were extended to other loading scenarios by Mindlin and Deresiewicz (1953). A significant generalization of these results has recently been discovered independently by Jäger (1997) and Ciavarella (1998a), who showed that the frictional traction distribution satisfying both equality and inequality conditions for *any* plane contact problem (not necessarily Hertzian) will consist of a superposition of the limiting friction distribution and an opposing distribution equal to the coefficient of friction multiplied by the normal contact pressure distribution at some smaller value of the normal load. Thus, as the tangential force is increased at constant normal force, the stick zone shrinks, passing monotonically through the same sequence of areas as the normal contact area passed through during the normal loading process. These results can be used to predict the size of the slip zone in conditions of fretting fatigue (Hills and Nowell, 1994; Szolwinski and Farris, 1996). One consequence of this result is that wear in the sliding regions due to an oscillating tangential load will not change the extent of the adhesive region, so that in the limit the contact is pure adhesive and a singularity develops in the normal traction at the edge of this region (Ciavarella and Hills, 1998). Results for more general loading scenarios have been established by Jäger (1998).

The results carry over to the general three-dimensional contact problem strictly only for the case where Poisson's ratio is zero (Ciavarella, 1998b) and where the direction of the tangential force does not change. In all other cases, the 'Cattaneo' traction distribution satisfies the limiting friction equations but predicts a mismatch between the direction of the traction and the direction of slip. However, similar conditions apply to the original Cattaneo and Mindlin solutions and the effects of this error have been shown to be small in particular cases. With this caveat, the results of (Ciavarella, 1998b) can probably be automatically extended to the general loading case using the arguments of Jäger (1997). The resulting 'Ciavarella–Jäger theorem' would be a very powerful tool in the understanding of frictional quasi-static and impact problems for half-spaces of similar materials.

Difficulties are encountered with both existence and uniqueness proofs for the static elastic contact problem with Coulomb friction (Kikuchi and Oden, 1988: Chapter 10). Some of these difficulties can be resolved by using a non-local friction law, which smooths the discontinuities inherent in the Coulomb law. However, the uniqueness proof still requires that the coefficient of friction be sufficiently small (Oden and Pires, 1983). Similar results for the quasi-static problem were obtained by Cocu et al. (1984). The question of non-uniqueness of quasi-static problems with Coulomb friction has been extensively studied by Klarbring (1984, 1990). In particular, he examined the behaviour of a simple two degree of freedom system involving an elastically-supported rigid body that can slide on a rigid plane or separate from it under the influence of applied forces. Klarbring showed that for sufficiently high coefficients of friction, loading conditions exist under which three different quasi-static states are possible – stick, slip in one direction and separation.

Martins and Oden (1987), Martins et al. (1992, 1994) generalized Klarbring's model to include viscous damping and showed that the solution is then always unique. In the limit of vanishing damping, their algorithm exhibits scenarios in which instantaneous jumps in position and state can occur when the limiting friction condition is exceeded in one direction. Similar conclusions were reached by Cho and Barber (1998) for a dynamic algorithm in the limit of vanishing mass. These results are significant in view of the fact that an existence theorem can be established with arbitrary coefficient of friction if the requirement of continuity of displacement is relaxed (Martins et al., 1992).

Many attempts have been made to modify the Coulomb friction law to avoid the difficulties of non-uniqueness, whilst retaining the essential physics of the process. An important motivation for doing this is to be able to formulate self-consistent finite element algorithms for frictional problems. One of the most studied alternatives is the *normal compliance law*, in which the tangential and normal tractions are defined as separate power law functions of a small relative normal displacement at the interface (Klarbring, 1990). Klarbring et al. (1989) and Andersson (1991) proved uniqueness for the normal compliance law subject to certain constraints on the parameters.

1.2. Dynamic instabilities

In practice, non-unique solutions to quasi-static frictional problems are usually associated with systems which can become 'locked' as a result of friction, such as a narrow angle wedge forced into a tapered slot. In such cases, transitions between states may involve dynamic instabilities. However, even the existence of a unique quasi-static frictional solution affords no guarantee of dynamic stability and many practical examples of frictionally-induced vibrations have been reported, including brake squeal and stick–slip vibrations (Ibrahim, 1994). The classical mechanism proposed to explain this behaviour involves dependence of friction coefficient on sliding speed or a difference between static and dynamic coefficient of friction, but Adams (1995) has shown that slow quasi-static slip between two elastic half-planes can be unstable even when f is constant. He also demonstrates the existence of a quasi-steady solution in which regions of stick and slip propagate at the interface (Adams, 1998). The ratio between the tangential and normal tractions in the stick zones is lower than f and hence the ratio $f*$ between the applied tangential and normal forces will also be less than f. Furthermore, Adams shows that $f*$ falls with increasing sliding speed, even though f is held constant. This speed dependence of friction coefficient is well known from experimental data, but of course there are many other possible mechanisms as well as that described here.

Cho and Barber (1999) investigated a three-dimensional extension of Klarbring's model in which the block has two degrees of freedom of sliding in the contact plane and showed that arbitrarily slow quasi-static sliding in certain directions can be unstable, even when f is sufficiently small for the quasi-static solution to be unique. The relationship between continuous and point mass models of instability for a specific friction law is discussed by Ionescu and Paumier (1994).

2. Self-similarity in indentation problems

Some strikingly simple but general results have been achieved making use of the concept of self-similarity in problems involving the indentation of a deformable half-space by a rigid body with a power law profile.

The method proceeds by examining the conditions under which the stress and displacement fields at various stages in the monotonic loading process can be mapped into each other with a load-dependent scalar multiplier. Spence (1973) showed that the indentation of an elastic half-space by a power law punch with Coulomb friction is self-similar and hence that the extent of the central adhesion zone maintains a constant ratio to the extent of the contact area. He was also able to show that the governing integral equations for the power law punch could be transformed into those for indentation by a flat punch and hence that this ratio was the same for all such punches. In particular, this enabled him to extend the classical axisymmetric Hertz solution to include the effect of interfacial friction (Spence, 1975).

More recently, several authors have shown that the same technique can be used in problems where the constitutive law is non-linear, provided this is of power law form. Hill et al. (1989) gave a self-similar solution for the plastic indentation of a fairly general power law material by a spherical indenter and were thereby able to give a rigorous theoretical explanation of the empirical relation between the applied load, the radius of the sphere and the radius of the indentation, known as Meyer's law. Borodich (1993) has made substantial contributions to the formalism of the self-similarity approach and has also shown that self-similarity is preserved when both material non-linearities and finite frictional effects are included.

Corresponding results for Vickers and Berkovich indentation, which involves a non-axisymmetric pyramidal indenter, were given by Giannakopoulos et al. (1994), Larsson et al. (1996) and Giannakopoulos and Larsson (1997). The same method was applied to creeping materials by Bower et al. (1993) and Storåkers et al. (1997), using results from Hill (1992). In all these cases, the indentation problem for the entire loading history is reduced to the solution of a single non-linear boundary value problem. More importantly, in this reduced problem the contact area is fixed, in contrast to a conventional incremental solution of the problem in which the contact area would increase with the load. This greatly facilitates the numerical solution of the reduced problem, since the finite element mesh can be chosen appropriately to the extent of the contact area and discretization errors as the contact area expands over the mesh are eliminated.

These studies are particularly useful in providing a rigorous mathematical foundation to the interpretation of data from indentation tests, which are becoming increasingly used as a method of measuring material constitutive behaviour (Tabor, 1986).

3. Elastic limit, shakedown, ratchetting and wear

Detailed solutions for the interior stress fields in elastic contact problems have been collected by Hills et al. (1993), with special reference to the probable location of initial failure, and the load at which yield is predicted. They consider Hertzian contacts under normal and tangential loading, including also careful studies of second order effects, such as those due to elastic dissimilarity if friction is present. Other geometries are rarely considered, although analytical solutions can be obtained for various other cases including the cone and the rigid flat punch. For the latter case, the elastic solution exhibits a square root singularity in normal traction at the corners, suggesting that LEFM criteria are appropriate (Giannakopoulos and Larsson, 1997). However, if the punch corners are rounded, the stresses are everywhere bounded and Ciavarella et al. (1998) have shown that a fairly small radius is sufficient to increase the strength of the contact significantly, particularly in the case of the axisymmetric flat punch (Ciavarella, 1999), for which rounding over 20% of the punch base is sufficient to render the contact stronger than a Hertzian contact with the same contact area and applied force. Thus, it is arguable that a conventional plasticity criterion is more appropriate than one based on singularity strengths, particularly in those geometries where the singularity is less than square root.

Above the elastic limit, contact problems generally exhibit a range of alternating loads within which the system will shake down — i.e. develop beneficial residual stresses which prevent further plastic deformation (Johnson, 1985: §9.2). As in conventional elastic–plastic problems, a lower bound to the shakedown limit can be found using Melan's theorem which states that if any self-equilibrating system of residual stresses can be found which would prevent yield, then the system will shakedown.

If a cyclic load above the shakedown limit is applied, the system may exhibit cyclic plasticity about a mean state, or progressive accumulation of plastic strain (ratchetting). It is not always easy to determine which of these situations will occur, but it is important to do so if possible, since failure in the first case is determined by low-cycle fatigue and in the latter by exhaustion of ductility or static plastic collapse, in which case failure is generally considerably more rapid. There is a rich literature on shakedown theorems, including a remarkable recent paper by Feng and Liu (1996) proving that, for a kinematic strain-hardening structure the only possible failure mode is cyclic plasticity and not ratchetting. Polizzotto (1997) derives shakedown theorems specialized for frictionless contact problems, i.e. considering the role of inequalities and variable contact area domain. In particular, it is proved that ratchetting collapse modes involving only normal displacements against a rigid obstacle are not possible.

In recent years, several ad hoc numerical methods have been proposed to compute the shakedown limit, generally using the concepts of upper and lower bounds to derive algorithms (see e.g. Ponter and Carter, 1997). For the special case of rolling and sliding contacts, a notable method is that due to Yu et al. (1996), which transforms the elastic–plastic problem for a linear–kinematic hardening material into a purely elastic problem and a residual problem with prescribed eigenstrains. The three-dimensional residual problem is then further reduced, using a method based on the purely elastic solution, into an elastic plane problem and an elastic anti-plane problem solved by standard FEM techniques.

The modelling of ratchetting is an extremely challenging problem, an extensive review of recent attempts being given by Ohno (1997). However, ratchetting forms the basis of a promising model of wear due to Kapoor and Johnson (1995). They showed that wear during erosive impact conditions occurs only when the impact contact stresses exceed the shakedown limit and the resulting wear takes the form of the extrusion of thin slivers of material which fracture to form wear debris. A similar mechanism is proposed for lubricated sliding, associated with the cyclic stresses due to asperity interaction (Kapoor et al., 1994).

4. Anisotropic materials

Interest in anisotropic materials originated with the properties of single crystals and highly distorted multi-granular materials, but has been reinvigorated by the extensive study of modern composites. The classical elastic contact problems associated with the names of Hertz and Boussinesq are not significantly more challenging than in the isotropic case. Two-dimensional problems can be formulated in complex variable terms analogous to Muskhelishvili's method for isotropic materials (Fan and Keer, 1994; Fan, 1996). In a related method due to Stroh (1958), linear transformations of the coordinates are found that reduce the governing equations to Laplacian form. This leads to a sextic algebraic equation for the transformation constants which generally cannot be solved in closed form (Head, 1979), but the full potential function solution in the transformed space is then of classical form.

The close parallels between isotropic and anisotropic problems can also be argued by reference to the form of the corresponding surface Green's functions, which here correspond to the physical problem of a concentrated force acting on the surface of a half-space. Similarity and equilibrium considerations dictate that surface displacements must vary logarithmically with distance from the force in the plane problem and it follows immediately that the contact pressure must have the same form as in the isotropic problem. For the corresponding three-dimensional problem of loading of a half-space by a point force, the surface displacement varies inversely with distance from the force, but now can have a fairly general dependence on angle at a given radius. However, Willis (1967) showed that the contact pressure distribution under a flat rigid elliptical punch is nonetheless unaffected by the anisotropy. In the Hertzian problem, the anisotropy affects the elipticity of the contact area, but Willis (1966) demonstrated that the Hertzian pressure distribution is retained and also proved that Galin's theorem, which defines the form of pressure distribution required to produce a general polynomial indentation inside an elliptical area, remains true for anisotropy.

The complete solution of the three-dimensional anisotropic Hertz problem requires that the angular variation of the surface Green's function be calculated. Willis gave a double Fourier transform solution to this problem, but various alternatives have been developed (see e.g. Ting, 1996). A remarkable result is that the required angular variation is proportional to the corresponding variation in compliance under a two-dimensional load as the anisotropic material is rotated under the load (Barber and Sturla, 1992; Ting, 1996). A more direct proof of this result can be obtained using Sobolev's transformation (Sveklo, 1964). An alternative numerical scheme based on the tensor notation of Barnett and Lothe (1973) and the Stroh formalism has been given by Vlassak and Nix (1994).

Considerable simplification is possible for the restricted case of transverse isotropy in planes parallel to the contact surface. The Green's function is then axisymmetric and results to a wide range of contact problems can conveniently be obtained by potential function methods (Fabrikant, 1989). Hanson (1992) has recently given a detailed treatment of the corresponding Hertzian problem, including the effects of tangential tractions due to sliding.

Most studies of anisotropic contact problems have restricted attention to the contact pressure distribution. This is probably justifiable for the general anisotropic contact problem (21 independent elastic constants), particularly in the important case of Hertzian contact, where the contact pressure keeps the same functional form and hence the value of the maximum pressure gives an indication of the severity of the loading.

The calculation of the full stress and displacement fields presents no major additional challenge, but failure theories for anisotropic materials are of such complexity that conclusions regarding contact failure and fatigue would only be possible for specific materials. However, it is remarkable that there are very few failure studies in the simpler case of orthotropic materials or even for transverse isotropy, where the Hertzian stress field is available in closed form.

5. Elastodynamic contact

When the loads applied to the contacting bodies move or change in time, the governing equations should be modified to include inertia terms. In most practical applications the accelerations and hence the inertia terms are small enough to be neglected, leading to the quasi-static formulation, but the full elastodynamic solution is of interest, particularly in cases involving impact or vibration. The conditions that must be satisfied to justify a quasi-static approximation are discussed by Johnson (1985) §11.4.

Analytical solutions of elastodynamic contact problems have largely been restricted to cases either of steady motion or self-similarity, both of which permit the acceleration terms to be replaced by spatial derivatives. Both problems fall into a broader class in which the coordinates defining the boundaries of the contact region are linear functions of time (Brock, 1993). The technique was first used by Bedding and Willis (1973) for the frictionless elastodynamic indentation of a general anisotropic elastic half space by an impacting rigid wedge or cone. Many problems of this class have since been solved by similar methods, a good review being given by Brock and Georgiadis (1994), who also give a solution to several self-similar indentation problems involving Coulomb friction.

In the case of steady motion, the resulting modified equations have the same form as those for an orthotropic material. Churilov (1977) used this result and the method of Sveklo (1964) to obtain a solution for the surface displacements due to a point force moving over the surface of a half-space at a speed below the shear wave speed. The analogy with anisotropic media shows that the elastodynamic Boussinesq and Hertz problems will have similar contact pressure distributions over the elliptical area to those in the quasi-static limit (Churilov, 1978). A more detailed study of the elastodynamic Hertz problem was given by Rahman (1996). The anisotropic analogy breaks down above the shear wave speed, because the elasticity matrix of the equivalent anisotropic material then ceases to be positive definite. However, the corresponding moving point force results were extended into this range by Barber (1996), using the Sobolev superposition directly on the elastodynamic equations.

The two-dimensional problem of steady motion of a rigid indenter over an elastodynamic half-plane was treated by Craggs and Roberts (1967). They found a well-behaved solution for speeds below the Rayleigh wave speed and above the dilatation wave speed, but no-one has since found a satisfactory solution for intermediate speeds. Some aspects of this paradox are discussed by Georgiadis and Barber (1993).

6. Effect of surface roughness

Real surfaces are rough on the microscopic scale and the effect of roughness on the contact process, particularly in sliding contact, forms the basis of most models of friction and wear. Contact is generally restricted to a number of microscopic 'actual contact areas' located near the peaks or 'asperities' of the rough surface. A common analytical technique is therefore to model the real surface as a statistical distribution of asperities of prescribed shape. The total load is then the sum of the individual loads on the contacting asperities, each of which is compressed a distance depending on its initial height.

In this field, an early breakthrough was made by Greenwood and Williamson (1966), who discovered that many important properties of the contact are almost independent of the details of the local asperity behaviour if the asperity height distribution is Gaussian. In the special case of an exponential distribution of identical asperities, they showed that the relations between total load, thermal and electrical contact conductance and total contact area are all linear, regardless of the constitutive law describing the contact process at the actual contact areas. More recent developments have therefore tended to concentrate on the description of the contacting surfaces as a stationary random process. Greenwood (1984) established relations between various treatments of this kind and discussed the relation between profilometer measurements and the properties of the surfaces.

Improvements in experimental methods have increased the bandwidth of surface profile measurements and revealed the existence of a hierarchy of scales up to the limits of experimental discrimination. For typical surfaces, the spectral density $P(\omega)$ exhibits a power law form $P(\omega)=C\omega^{-n}$ at high frequencies ω, but falls below this value at low frequencies. This low frequency attenuation could be associated with the finite length of the body, but is probably more aptly attributed to the 'success' of the manufacturing process. After all, if all surfaces were essentially random processes on all length scales, it would be a severe indictment of our ability to manufacture a surface of a prescribed shape!

The existence of an apparently inexhaustible sequence of smaller and smaller length scales is an embarrassment to asperity model theories, because the definition of an asperity is scale dependent. Thus, whereas with a large sampling interval, we see only a few asperities of large radius of curvature, as the sampling interval decreases we see more and more asperities of smaller radius. Some, but not all, properties of the contact are preserved as the truncation limit is extended. For example, the relationship between the total actual area of contact A and the normal load P is predicted to be almost linear ($A=CP$) at all length scales, but the constant of proportionality C decreases with sampling interval. More seriously, Greenwood and Williamson's 'plasticity index' which defines the extent of plastic deformation to be anticipated in elastic–plastic asperity contacts, appears to increase without limit as the sampling length is reduced, showing that the smaller scale asperities will always deform plastically.

These scale effects and the power law spectral density behaviour are strong indicators that a fractal description of the surface and the contact process would be more appropriate (Majumdar and Bhushan, 1995). Indeed, in a truly prescient paper, Archard (1957) suggested a model of rough surfaces in which a progression of smaller hemispherical asperities were superposed on a larger scale, which in the limit defines what we would now describe as a fractal surface. Archard used his model to establish that the resulting total actual contact area would be proportional to the applied load despite the non-linearity of the Hertzian contact equations.

The fractal properties of rough surfaces can be exposed by plotting various statistical measures of the profile on a logarithmic scale, in which case a true fractal will plot as a straight line. Majumdar and Bhushan (1995) give an extensive review of both experimental and theoretical aspects of the subject and recommend the use of the structure function

$$S(\tau)=\frac{1}{L}\int_0^L[z(x+\tau)-z(x)]^2\,\mathrm{d}x,$$

where $z(x)$ is the height of the surface at position x and L is the length of the sample. If the surface is fractal, this will be a power law relation of the form

$$S(\tau)=(G^{D-1}\tau^{2-D})^2,$$

where D is defined as the *fractal dimension* and G is a constant with units of length related to the amplitude of the roughness. Typical rough surface profiles are found to lie in the range $1<D<1.5$. The advantage of the fractal description is that it eliminates the implied truncation at small length scales by assuming that the same power law behaviour continues without limit to $\tau=0$. In the same way, the spectral density $P(\omega)$ will be assumed to have the same power law form for arbitrarily large ω. A recent review of the mathematics of fractal characterization and contact is given by Borodich and Onishchenko (1997). Wang and Komvopoulos (1995) used similar techniques to predict the distribution of surface temperature during sliding.

Majumdar and Bhushan (1991) developed a theory of contact for fractal surfaces based on the assumption that the distribution of actual contact area sizes would be similar to that of the 'islands' generated by cutting through the surface at a constant height z. They then obtained curvatures for the asperities so defined from random process theory and predicted the distribution of forces required to deform the asperities to the specified depth. Borri-Brunetto et al. (1998a) gave a more direct treatment of the fractal contact problem by first creating a finite realization of a fractal surface with the required properties and then using a numerical method to solve the resulting elastic contact problem at various levels of spatial discretization. With a coarse discretization, they obtained a few large actual contact areas, but as the grid was refined, these broke up progressively into clusters of smaller and smaller areas and the total area of actual contact decreased. This suggests that in the fractal limit the contact will consist of an infinite number of infinitesimal contact areas of total area zero. This agrees with Archard's model, but contrasts with that of Majumdar and Bhushan, which seems to predict a fractal distribution of areas of finite size.

Borri-Brunetto et al. (1998a) show that a contact area at any given scale resolves into a cluster of areas at the next smaller scale. With such clustering, it is essential to include the asperity interaction effect, which is the displacement produced at one asperity due to the contact force at another. Similar considerations apply to the prediction of electrical and thermal contact resistance. In fact, a sufficiently dense cluster of contact areas acts essentially the same way as an extended contact area of the same overall extent, which goes some way towards explaining why fairly coarsely truncated asperity model theories give acceptable predictions.

If the contacting surfaces are generated by fracture or by solidification against a mould, the profiles will conform, leading to a larger actual contact area than that between two similar but randomly oriented surfaces. In such cases, tangential displacement reduces the contact area significantly and forces the contacting bodies apart (Borri-Brunetto et al., 1998b).

7. Thermoelastic contact

If the temperatures of two contacting bodies are non-uniform, the resulting thermoelastic deformations will affect the contact pressure distribution and possibly the extent of the contact area. These changes in turn will generally affect the boundary conditions of the heat conduction problem, leading to a coupled thermomechanical problem, even when uncoupled thermoelasticity is assumed for the field equations.

The separate existence and uniqueness theorems for heat conduction and frictionless contact do not apply to this coupled problem and there are now many documented counter examples to such theorems with superficially reasonable physical boundary conditions (Dundurs and Comninou, 1976). Duvaut (1979) established an existence theorem for a boundary condition in which the contact resistance at the interface between the bodies is a monotonically decreasing function of pressure and was also able to prove uniqueness if the gradient of this function was sufficiently small. Consideration of the roughness of the contacting surfaces leads us to expect such a pressure-dependent resistance (see above), but experiments show that it is extremely sensitive to pressure and hence that the conditions for uniqueness will often not be met in practice.

Steady-state solutions, even when unique, may not be stable. Barber and Zhang (1988) demonstrated this behaviour for a simple one-dimensional system consisting of two one-dimensional rods contacting on an end face where there is a pressure-dependent thermal contact resistance. Existence and uniqueness theorems for problems of this class were considered by Andrews et al. (1994). When there are several steady-state solutions, one of them is generally stable, but when the solution is unique but unstable, the long-time transient behaviour of the system consists of an oscillation between contact and separation. There is experimental evidence of bi-stable or unstable systems driven by this coupling process (Srinivasan and France, 1985).

Thermal contact resistance between the casting and the mold has a major influence on solidification processes (Pehlke et al., 1973) and hence on the evolution of the metallurgical structure of the casting. Richmond and Huang (1977) showed that thermomechanical coupling can cause an initial perturbation in temperature at the mold surface to become unstable, resulting in extremely non-uniform solidification and even remelting or recrystallization of material near the casting surface. Yigit (1998) used a perturbation method to predict the evolution of this process for the solidification of a pure metal against a planar mould. A related problem for alloy solidification is discussed by Hector et al. (1996).

7.1. Frictional heating

One of the most technologically important areas involving thermoelastic contact is that in which the thermal problem is driven by the frictional heat generated during sliding. This coupled process is susceptible to thermoelastic instability (TEI) if the sliding speed is sufficiently high. Above the critical speed, a nominally uniform pressure distribution is unstable, giving way to localisation of load and heat generation and hence to hot spots at the sliding interface (Barber, 1969). These in turn can cause material damage and wear and are also a source of undesirable frictional vibrations (Lee and Dinwiddie, 1998). Similar problems arise in the sliding contact of electrical brushes, where they are complicated by electrical resistance heating (Lu and Bryant, 1994). Burton et al. (1973) used a perturbation method to investigate the stability of contact between two sliding half planes. The system is linearized about the uniform pressure state and perturbations are sought which can grow exponentially with time. This method has since been used for many other geometries as well as to investigate the stability of static contact (Barber, 1987). More recently, Yeo and Barber (1996) and Du et al. (1997) have shown how Burton's perturbation method may be implemented numerically, leading to an eigenvalue problem to determine the stability boundary. A more direct approach to the investigation of TEI is to use a numerical method to solve the coupled transient thermoelastic contact problem in time (Zagrodzki, 1990; Johansson, 1993). This method is extremely computer-intensive, but it has the advantage that it is readily adapted to practical loading cycles, which is of importance in the application to transmission clutches which experience intense periods of operation with rapidly varying sliding speed. Analytical methods of treating some idealized problems with variable sliding speed are discussed by Olesiak et al. (1997) and Yevtushenko and Chapovska (1997).

8. Conclusions

Contact mechanics, though classical, continues to be a subject of lively interest from many different perspectives, including mathematics, applied mechanics, numerical analysis, surface physics and experimental methods. It generates a seemingly inexhaustible sequence of commercially important and challenging problems, possibly because of the complexity of physical effects that can occur at and in the vicinity of contacting surfaces.

Length considerations necessarily restrict the coverage of the present review, so the authors have chosen to concentrate on topics in which there have been significant recent developments and which are close to their own research interests. Significant areas unfortunately excluded on these grounds include adhesion and viscoelastic materials.

Attempts to identify the challenging problems of the future tend to say more about the lack of foresight of the forecasters than about the development of science. However, we feel confident in predicting significant developments in the fractal characterization of the contact of rough surfaces and in the general area of thermoelastic/plastic contact. There are also many interesting developments at the interface between dynamics and contact mechanics, particularly problems involving friction, which have extremely broad industrial and scientific applications. Finally, contact mechanics is coming to have an increasing presence in the study of fracture, through crack closure during fatigue crack propagation and mode II dominated failure in many composite materials applications, where friction is also a factor.

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